The construction of the q-analogues of the harmonic oscillator operators from ordinary oscillator operators

1990 J. Phys. A: Math. Gen. 23 L821

(http://iopscience.iop.org/0305-4470/23/16/008)

J. Phys. A: Math. Gen. **23** (1990) L821–L825. Printed in the UK

LETTER TO THE EDITOR

The construction of the q-analogues of the harmonic oscillator operators from ordinary oscillator operators

Xing-Chang Song

China Center of Advanced Science and Technology (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
and
Institute of Theoretical Physics, Academia Sinica, PO Box 2735, Beijing 100080, People's Republic of China
and
Department of Physics, Peking University, Beijing 100871, People's Republic of China

Received 1 May 1990

Abstract. The q-analogues of Boson operators are constructed from ordinary Boson operators by an embedding approach. Accordingly the q-deformed quantum algebras $SU(n)_q$ can be obtained straightforwardly from corresponding universal enveloping algebras of Lie algebra $SU(n)$.

In recent years there has been a great deal of interest in the study of the quantum group [1]. More recently, a new realization of the quantum group $SU(2)_q$ has been written down by introducing a q-analogue of the quantum harmonic oscillator [2, 3]. In this letter we will propose a method to construct the q-analogue of the harmonic oscillator boson operators in terms of the ordinary ones. This construction provides an underlying understanding of the relation between a quantum group $\tilde{G}$ and its corresponding Lie algebra of the ordinary group G.

The quantum group $SU(2)_q$ is a q-deformation of the Lie algebra of SU(2). It is generated by operators $\tilde{J}_+$, $\tilde{J}_-$ and $\tilde{J}_0$ obeying the relations

$$[\tilde{J}_0, \tilde{J}_\pm] = \pm\tilde{J}_\pm \tag{1}$$

$$[\tilde{J}_+, \tilde{J}_-] = [2\tilde{J}_0] \tag{2}$$

where for given x

$$[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}} \equiv \frac{\sinh \eta x}{\sinh \eta} \tag{3}$$

and $q = e^\eta$ is a parameter, usually taken to be real for simplicity. In the limit $q \to 1$ ($\eta \to 0$), $SU(2)_q$ reduces to the ordinary Lie algebra SU(2) generated by $J_\pm$ and J_0 satisfying

$$[J_0, J_\pm] = \pm J_\pm \tag{4}$$

$$[J_+, J_-] = 2J_0 \tag{5}$$

0305-4470/90/160821+05\$03.50

which can be put into the Jordan–Schwinger form by means of a pair of independent boson operators:

$$J_+ = a_1^+ a_2 \qquad J_- = a_2^+ a_1 \qquad 2J_0 = a_1^+ a_1 - a_2^+ a_2 \tag{6}$$

where

$$[a_i, a_j^+] = \delta_{ij} \tag{7}$$

with all other brackets vanishing. Or, by introducing the number operator $N_i \equiv a_i^+ a_i$, the commutation relations read

$$[N_i, a_j^+] = a_j^+ \delta_{ji} \qquad [N_i, a_j] = -a_j \delta_{ij}. \tag{8}$$

It is well known that the whole Hilbert space of the harmonic oscillator is constructed as follows. The vacuum state $|0\rangle$ defined as

$$a|0\rangle = 0 \tag{9}$$

is the lowest eigenstate of the number operator N

$$N|0\rangle = 0|0\rangle. \tag{10}$$

The n-quanta eigenstates $|n\rangle$ given by

$$|n\rangle = \frac{(a^+)^n}{(n!)^{1/2}}|0\rangle \tag{11}$$

are orthonormal, with the properties

$$a^+|n\rangle = (n+1)^{1/2}|n+1\rangle \tag{12}$$

$$a|n\rangle = n^{1/2}|n-1\rangle \tag{13}$$

$$N|n\rangle = n|n\rangle \tag{14}$$

and the angular momentum eigenstates $|j, m\rangle$ are now described by

$$|j, m\rangle = \frac{(a_1^+)^{j+m}}{\sqrt{(j+m)!}} \frac{(a_2^+)^{j-m}}{\sqrt{(j-m)!}}|0\rangle \equiv |n_1 = j+m\rangle \otimes |n_2 = j-m\rangle \tag{15}$$

with the properties following on from (6), (12) and (13)

$$J_0|j, m\rangle = m|j, m\rangle \tag{16}$$

$$J_+|j, m\rangle = \sqrt{(j-m)(j+m+1)}|j, m+1\rangle \tag{17}$$

$$J_-|j, m\rangle = \sqrt{(j+m)(j-m+1)}|j, m-1\rangle \tag{18}$$

as required.

It has been pointed out that [2, 3], in a similar way, the quantum $SU(2)_q$ algebra relations (1) and (2) can be realized by introducing a q-analogue to the harmonic oscillator with q-creation operator $\tilde{a}^+$, q-annihilation operator $\tilde{a}$ and number operator $\tilde{N}$ satisfying

$$[\tilde{N}, \tilde{a}^+] = \tilde{a}^+ \qquad [\tilde{N}, \tilde{a}] = -\tilde{a} \tag{19}$$

$$[\tilde{a}, \tilde{a}^+] = [\tilde{N}+1] - [\tilde{N}]. \tag{20}$$

Then starting from a q-boson vacuum $|\tilde{0}\rangle$ defined by $\tilde{a}|\tilde{0}\rangle = 0$ one can obtain the n-quanta eigenstates

$$|\tilde{n}\rangle = \frac{(\tilde{a}^+)^n}{\sqrt{[n]!}}|\tilde{0}\rangle \tag{21}$$

with

$$\tilde{N}|\tilde{n}\rangle = n|\tilde{n}\rangle \tag{22}$$

$$\tilde{a}^+|\tilde{n}\rangle = [n+1]^{1/2}|\widetilde{n+1}\rangle \tag{23}$$

$$\tilde{a}|\tilde{n}\rangle = [n]^{1/2} \qquad |\widetilde{n-1}\rangle \tag{24}$$

where $[n]! = [n][n-1]\ldots[2][1]$. Repeating the procedure given above, the eigenstates of the q-deformed angular momentum can now be described by two commuting oscillators [2, 3]

$$|\widetilde{j,m}\rangle = \frac{(\tilde{a}_1^+)^{j+m}}{\sqrt{[j+m]!}}\frac{(\tilde{a}_2^+)^{j-m}}{\sqrt{[j-m]!}}|\tilde{0}\rangle. \tag{25}$$

Then, upon these states, the $\widetilde{\mathrm{SU}(2)}_q$ algebraic relations (1) and (2) can be satisfied by the following identification

$$\tilde{J}_+ = \tilde{a}_1^+\tilde{a}_2 \qquad \tilde{J}_- = \tilde{a}_2^+\tilde{a}_1 \qquad 2\tilde{J}_0 = \tilde{N}_1 - \tilde{N}_2. \tag{26}$$

One easily verifies that

$$\tilde{J}_\pm|\widetilde{j,m}\rangle = \sqrt{[j\mp m][j\pm m+1]}\,|\widetilde{j, m\pm 1}\rangle \tag{27}$$

$$\tilde{J}_0|\widetilde{j,m}\rangle = m|\widetilde{j,m}\rangle \tag{28}$$

which coincide with Jimbo's representation [4]. In deriving (2), use has been made of the important relation

$$[n_1][n_2+1] - [n_2][n_1+1] = [n_1 - n_2]. \tag{29}$$

At this stage, the relation (2) holds only on ket vectors $|\widetilde{j,m}\rangle$ as is stressed by Biedenharn [2].

Now it is essential to notice that the q-analogue operators $\tilde{a}_i^+$, $\tilde{a}_i$ and $\tilde{N}_i$ can be constructed by the ordinary boson operators a_i^+, a_i and N_i in the following way:

$$\tilde{N}_i = N_i = a_i^+ a_i \tag{30}$$

$$\tilde{a}_i = a_i\sqrt{\frac{[N_i]}{N_i}} = \sqrt{\frac{[N_i+1]}{N_i+1}}\,a_i = a_i\sqrt{\frac{\sinh \eta N_i}{N_i \sinh \eta}} \tag{31}$$

$$\tilde{a}_i^+ = \sqrt{\frac{[N_i]}{N_i}}\,a_i^+ = a_i^+\sqrt{\frac{[N_i+1]}{N_i+1}} = \sqrt{\frac{\sinh \eta N_i}{N_i \sinh \eta}}\,a_i^+. \tag{32}$$

Then it follows that

$$[N_i, \tilde{a}_i] = [N_i, a_i]\sqrt{\frac{[N_i]}{N_i}} = -a_i\sqrt{\frac{[N_i]}{N_i}} = -\tilde{a}_i \tag{33}$$

$$[N_i, \tilde{a}_i^+] = [N_i, a_i^+]\sqrt{\frac{[N_i+1]}{N_i+1}} = a_i^+\sqrt{\frac{[N_i+1]}{N_i+1}} = \tilde{a}_i^+ \tag{34}$$

and

$$\tilde{a}_i^+ \tilde{a}_i = \sqrt{\frac{[N_i]}{N_i}}\, a_i^+ a_i \sqrt{\frac{[N_i]}{N_i}} = [N_i] \tag{35}$$

$$\tilde{a}_i \tilde{a}_i^+ = \sqrt{\frac{[N_i+1]}{N_i+1}}\, a_i a_i^+ \sqrt{\frac{[N_i+1]}{N_i+1}} = [N_i+1]. \tag{36}$$

These reproduce relations in (19) and (20). Then $\widetilde{\mathrm{SU}(2)}_q$ follows relations (1) and (2) immediately from the identification (26). Notice that in the above prescription the identifications (31), (32) and (26), as well as the $\widetilde{\mathrm{SU}(2)}_q$ algebraic relations (1) and (2) are all operator formulae here. They do close abstractly, but do not merely stand on particular ket vectors.

In the above construction procedure, when one goes from the ordinary oscillator into the q-deformed one, the operators a and a^+ are changed to their tilded counterparts $\tilde{a}$ and $\tilde{a}^+$ but the number operator N remains the same. So the vacuum state $|0\rangle$ for ordinary oscillator remains as the q-deformed vacuum state

$$|\tilde{0}\rangle = |0\rangle \tag{37}$$

and from (16) one sees that

$$(\tilde{a}^+)^n = (a^+)^n \sqrt{\frac{[N+n]!}{[N]!}\frac{N!}{(N+n)!}}. \tag{38}$$

This gives

$$(\tilde{a}^+)^n|0\rangle = (a^+)^n \sqrt{\frac{[n]!}{n!}}\,|0\rangle \tag{39}$$

or equivalently

$$|\tilde{n}\rangle = |n\rangle. \tag{40}$$

This implies that the same set of eigenvectors $|n\rangle$ expands the whole Hilbert space both for the ordinary harmonic oscillator and for its q-analogue. For the same reason $\widetilde{|j, m\rangle} = |j, m\rangle$, and the same set of eigenvectors $|j, m\rangle$ describes the whole spectrum both for Lie algebra SU(2) and for its q-analogue $\widetilde{\mathrm{SU}(2)}_q$. As a special case, for fixed $j = 0, \frac{1}{2}, 1, \ldots$ with m running by integer steps over the range $j \geqslant m \geqslant -j$ the set of eigenstates $|j, m\rangle$ defines finite-dimensional $(2j+1)$ unitary irreps of SU(2), as well as irreps of $\widetilde{\mathrm{SU}(2)}_q$. For this special case, the fact mentioned above coincides with the embedding of $\widetilde{\mathrm{SU}(2)}_q$ into the universal enveloping algebra of SU(2) originally discussed by Jimbo [4] in an asymmetric way. Recently a symmetric embedding scheme of $\widetilde{\mathrm{SU}(n)}_q$ into $\mathrm{SU}(n)$ is given [5], which is closely relevant to the result presented here.

An interesting deformation occurs when q takes the special values of being the kth root of unity, i.e.

$$q = \mathrm{e}^{2\pi i/k} \qquad k \in Z. \tag{41}$$

In this case $[n+k] = [n]$, and the infinite series $[n]$ is truncated leaving only the subset $[1], [2], \ldots, [k] = [0]$ allowed. For example, when $q = -1$, only two values are allowed:

$$[\text{even}] = [0] = 0 \tag{42}$$

$$[\text{odd}] = [1] = 1. \tag{43}$$

Then one sees from (35), (36)

$$\tilde{a}^+\tilde{a}+\tilde{a}\tilde{a}^+=[N]+[N+1]=1 \tag{44}$$

which implies the oscillator is fermionic.

The construction procedure mentioned above can be easily generalized to the $\widetilde{\mathrm{SU}(n)}_q$ case. As a simplest example for SU(3), three different kinds of independent operators a_i, a_i^+ are introduced with $[a_i, a_j^+]=\delta_{ij}$. Then the SU(3) generators can be identified as

$$H_1=a_1^+a_1-a_2^+a_2=N_1-N_2 \qquad H_2=a_2^+a_2-a_3^+a_3=N_2-N_3 \tag{45}$$

$$E_{12}=a_1^+a_2 \qquad E_{23}=a_2^+a_3 \qquad E_{13}=a_1^+a_3 \tag{46}$$

$$E_{21}=a_2^+a_1 \qquad E_{32}=a_3^+a_2 \qquad E_{31}=a_3^+a_1 \tag{47}$$

and the commutation relations follow at once

$$[H_1, E_{12}]=2E_{12} \qquad [H_1, E_{23}]=-E_{23} \qquad [H_1, E_{13}]=E_{13} \tag{48}$$

$$[H_2, E_{12}]=-E_{12} \qquad [H_2, E_{23}]=2E_{23} \qquad [H_2, E_{13}]=E_{13} \tag{49}$$

$$[E_{12}, E_{21}]=H_1 \qquad [E_{23}, E_{32}]=H_2 \tag{50}$$

$$[E_{12}, E_{23}]=E_{13} \tag{51}$$

etc. The q-analogue $\widetilde{\mathrm{SU}(3)}_q$ can be realized by writing $\tilde{a}_i$, $\tilde{a}_i^+$ as in (31) and (32):

$$\tilde{H}_1=H_1=N_1-N_2 \qquad \tilde{H}_2=H_2=N_2-N_3 \tag{52}$$

$$\tilde{E}_{12}=\tilde{a}_1^+\tilde{a}_2 \qquad \tilde{E}_{23}=\tilde{a}_2^+\tilde{a}_3 \qquad \tilde{E}_{13}=\tilde{a}_1^+\tilde{a}_3 \tag{53}$$

$$\tilde{E}_{21}=\tilde{a}_2^+\tilde{a}_1 \qquad \tilde{E}_{32}=\tilde{a}_3^+\tilde{a}_2 \qquad \tilde{E}_{31}=\tilde{a}_3^+\tilde{a}_1 \tag{54}$$

and the algebraic relations read

$$[H_1, \tilde{E}_{12}]=2\tilde{E}_{12} \qquad [H_1, \tilde{E}_{23}]=-\tilde{E}_{23} \qquad [H_1, \tilde{E}_{13}]=\tilde{E}_{13} \tag{55}$$

$$[H_2, \tilde{E}_{12}]=-\tilde{E}_{12} \qquad [H_2, \tilde{E}_{23}]=2\tilde{E}_{23} \qquad [H_2, \tilde{E}_{13}]=\tilde{E}_{13} \tag{56}$$

$$[\tilde{E}_{12}, \tilde{E}_{21}]=[H_1] \qquad [H_{23}, E_{32}]=[H_2] \tag{57}$$

$$[\tilde{E}_{12}, \tilde{E}_{23}]=\tilde{E}_{13}([N_2+1]-[N_2]) \tag{58}$$

etc, which are equivalent to those given by Jimbo [4] (in his notation $\hat{E}_{12}=\tilde{E}_{12}$, $\hat{E}_{23}=\tilde{E}_{23}$ but $\hat{E}_{13}=q^{-N_2}\tilde{E}_{13}$).

This work was supported in part by the National Natural Science Foundation of China. Helpful discussions with Professors H Y Guo and B Y Hou are acknowledged with thanks.

References

[1] de Vega H J 1989 *Int. J. Mod. Phys.* A **4** 2371
[2] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[3] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
Ng YJ 1990 *J. Phys. A: Math. Gen.* **23** 1023
[4] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Commun. Math. Phys.* **102** 537
[5] Hou B Y, Shi K J, Song X C and Wang P 1990 *Preprint* Northwest University, Xian, China NWUIMP-(89)-620, to appear in *Commun. Theor. Phys.*